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AN 'ASSUMED DEVIATORIC STRESS–PRESSURE–VELOCITY' MIXED FINITE ELEMENT METHOD FOR UNSTEADY, CONVECTIVE, INCOMPRESSIBLE VISCOUS FLOW: PART I: THEORETICAL DEVELOPMENT*

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SUMMARY

A formulation of a mixed finite element method for the analysis of unsteady, convective, incompressible viscous flow is presented in which: (i) the deviatoric-stress, pressure, and velocity are discretized in each element, (ii) the deviatoric stress and pressure are subject to the constraint of the homogeneous momentum balance condition in each element, *a priori*, (iii) the convective acceleration is treated by the conventional Galerkin approach, (iv) the finite element system of equations involves only the constant term of the pressure field (which can otherwise be an arbitrary polynomial) in each element, in addition to the nodal velocities, and (v) all integrations are performed by the necessary order quadrature rules. A fundamental analysis of the stability of the numerical scheme is presented. The method is easily applicable to 3-dimensional problems. However, solutions to several problems of 2-dimensional Navier–Stokes' flow, and their comparisons with available solutions in terms of accuracy and efficiency, are discussed in detail in Part II of this paper.

KEY WORDS Mixed Method Assumed Deviatoric Stress Galerkin Formulation

INTRODUCTION

Development of efficient and accurate finite element methods (FEM) for solving the Navier–Stokes equations governing unsteady, convective, incompressible, viscous fluid flow, has been the subject of intense scrutiny by several researchers over the past 8 years or so. The FEM developed so far can be labeled as 'semi-discrete' in nature–a finite element discretization in space and a temporal finite-difference approximation to treat the unsteady aspect of the flow.

The main issues germane to the development of successful semi-discrete approximations for Navier–Stokes unsteady flows can be broadly identified as: (i) treatment of the kinematic constraint of incompressibility, (ii) proper treatment of the operator, which represents the convective acceleration of the fluid particle, and whose Frechet derivative is unsymmetric, (iii) proper algorithms to solve the (unsymmetric) non-linear algebraic equations which arise for convection dominated flows under steady-state conditions, and (iv) proper algorithms for

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integrating the first-order (non-linear) ordinary differential equations which arise for unsteady, convection dominated flows. The literature contains many works which present diverse viewpoints in addressing each of these issues. Without embarking on an exhaustive survey of this still burgeoning literature, we shall mention, in the following, only a few representative works, which provide alternative treatments of each of the above four issues, which contain additional references to other important pertinent literature, and which are pertinent to the present work for the purpose of contrast and comparison.

In the treatment of incompressibility, the existing literature provides the following alteratives: (a) the so-called 'primitive-variable' or the assumed 'velocity-pressure' mixed formulation. This method has been extensively studied by Gresho, Sani, Lee, Upson, Chan, and Leone,[1–5] Olson and Tuann,[6] Gartling, Nickell and Tanner,[7] and Donea *et al.*[8] In this method, the hydrostatic pressure acts as a Lagrange multiplier to enforce the incompressibility constraint and is retained as an unknown vector in the global finite element system of equations. In this approach, in general, the basis functions for the assumed element velocities are one degree higher than those for the assumed element pressure.[4,5] In this method, in general, the velocity solutions obtained are much better than those for pressure, even though certain 'cures' and 'smoothing techniques' for generating 'good' pressure solutions have been suggested.[2,3] (b) The 'selective-reduced-integration-penalty methods' (SRIP). In this approach, the constitutive equation is modified through the introduction of a penalty parameter, and the hydrostatic pressure is eliminated *ab initio* from the formulation, but is computed by post-processing the obtained velocity solution. This elimination of pressure, which results in a smaller system of equations than in approach (a) above has been proclaimed by many to be one of its major advantages. This method has been extensively studied by Hughes, Brooks, Taylor, Tezduyar, Liu and Levy,[9–11] Heinrich and Marshall,[12] Bercovier and Engelman,[13] Nakazawa and Zienkiewicz[14,15] and Reddy,[16,17] among others. Recent theoretical investigations of these methods by Oden, Kikuchi, Song, and Jacquotte[18–21]—through studies of the so-called LBB conditions governing the SRIP methods—indicate that some of the SRIP methods are, in fact, unstable. Attempts have been made at 'averaging' or 'filtering' the pressures to stabilize these methods (see Reference 20 and the References therein) but the methods are, in general, still sensitive to singularities and distortions of the mesh. These drawbacks notwithstanding, there are some RIP methods which work highly satisfactorily for certain problems. It should be mentioned that Malkus and Hughes[22] have discussed the 'equivalence' between the above two methods, namely, the primitive variable mixed and SRIP methods, under certain circumstances. It was noted[22] that to each mixed element of the primitive variable type, with continuous velocities and discontinuous pressure fields, there corresponds an element and a reduced integration scheme in the penalty function formulation.

In the treatment of the operator of the convective acceleration term, the following approaches have been suggested in the literature: (a) the use of the standard Galerkin method, wherein the trial (or assumed basis functions for velocities) functions are the same as the test (or the weight functions in the weighted residual formulations) functions. This has been used by Gresho, Leone, Lee and Sani, Upson and Chan,[1–5] Donea *et al.*[8] and Reddy.[16] This technique may result in oscillations when the streamwise gradients become too large to be resolved by the mesh and, often, an appropriate solution to the wiggle problem is the use of selective grid refinement.[4] (b) The use of the so-called Petrov–Galerkin method, wherein the trial and test functions are different. This method has been studied by Heinrich and Zienkiewicz,[23] and Hughes *et al.*[9,11] who first introduced the so-called quadrature upwinding scheme which was found to exhibit cross-wind diffusion; Later, to remedy this, they introduced the so-called stream-line upwinding. However, it appears[4] that the upwind schemes, in general, effectively reduce the local Reynolds number and thereby generate deceptively smooth and often inaccurate results for the actual Reynolds number on hand. The 'upwind' finite element scheme has been earlier suggested[24] as a means of coping with second-order differential equations with significant first derivatives.

As for the issue of solving the unsymmetric non-linear equations for steady flow, the following techniques exist: (a) direct iteration, (b) Newton–Raphson or (c) the so called quasi-Newton or variable metric methods. The literature on this is quite exhaustive. We restrict ourselves to the techniques used in the present paper; those belonging to category (c) above. The details of these techniques can be found in the works of Dennis and More,[25] Engelman, Strang and Bathe,[26] and Geradin, Idelsohn and Hogge.[27] Finally, the literature on time-integration of the first-order, ordinary non-linear differential equations for unsteady flow is also exhaustive. Here we cite the representative works which are explicit, 'linearly explicit', or fractional step methods,[1,9,8] respectively, with further differences arising due to either consistent or lumped masses being employed.

With the above perspective, we now introduce the presently proposed method. Here, in summary: (i) the incompressibility constraint is treated through a mixed formulation based on assumed 'deviatoric stress-velocity-hydrostatic pressure' in each element. The present mixed method is thus radically different from the 'primitive variable' mixed method discussed earlier; (ii) the convective acceleration term is treated through a conventional Galerkin method. It should be remembered, however, that in the present formulation the momentum balance condition involves stresses directly along with the convective acceleration (in terms of spatial velocities and their gradients). The first derivatives of both the stress and velocity fields occur in the momentum equation, and both the stress and velocity are directly approximated in each element in the present method; (iii) the steady-state non-linear algebraic equations are solved by a quasi-Newton method with Broyden update, and (iv) the unsteady equations are solved by a 'linearly implicit' algorithm. The main object of our present paper is to discuss only items (i) and (ii) above—the distinguishing features of the present mixed method—whereas items (iii) and (iv) and variations thereof can be subjected of an independent study, irrespective of the spatial discretization scheme used.

Thus, in the present method, all the three fields—the deviatoric stress, the hydrostatic stress, and the velocity—are discretized in each element. In the present approach, the deviatoric fluid stresses and the hydrostatic pressure are assumed in each element so as to satisfy the homogeneous part of the linear momentum balance condition (the angular momentum balance condition being met by the symmetry of the chosen stress tensor). In addition, we assume a C° continuous velocity field over each element. Even though the hydrostatic pressure is an *arbitrary polynomial* in each element, *only the constant term* of this polynomial becomes a solution variable, in addition to the nodal velocities, in the global finite element system of equations. Thus, *for a given finite element mesh*, the present method results in a global system of equations which is larger than that in the SRIP methods[9–11] by only the number of elements used in the discretization. This additional computing becomes 'trivial' in the case of three-dimensional flow modelling. Moreover, we find that even this small additional computing is more than offset by the *excellent accuracy*, and *direct solution*, of *both* the velocity and pressure fields. Also, the present mixed method is found to be not only more accurate but also efficient as compared to the primitive variable mixed method.[4,12] Also in the present method, no selective reduced integration is employed: all integrations are performed with the necessary order quadrature rules.

The present mixed method for unsteady convection dominated Navier–Stokes flows reduces to a hybrid method for treating creeping 'Stokes' flows presented earlier by Bratianu, Ying, Yang, and Atluri.[28–31] For the Stokes flow, not only a demonstration of the versatility of the present approach,[28] but also a study of stability and convergence of the method[29] were presented.

In Part I of the current paper we present, in order: (i) the formulation of the semi-discrete approximation scheme using a mixed finite element scheme based on the assumed 'deviatoric stress-pressure-velocity' fields in each element, (ii) a study of the stability of the present mixed scheme, and (iii) some details of solutions of the non-linear algebraic and first-order differential equations. Specific details of formulation of certain 2-dimensional elements are given in Appendix I. A detailed comparison of the present mixed method with the standard 'velocity-pressure' mixed method is made in Appendix II. A detailed account of the computed solutions for various problems, and their comparison with those generated by the cited alternative methods, are given in Part II of this paper.

MIXED FINITE ELEMENT METHOD BASED ON ASSUMED (σ'_{ij}, p, v) FOR INCOMPRESSIBLE VISCOUS FLOW

Here we consider unsteady flow, at moderate to high Reynolds number, of an incompressible viscous fluid in a domain V with spatial co-ordinates x_i. We use the notation: ρ the fluid density; $\bar{F}_i$ the body forces (excluding inertia) per unit mass; σ_{ij} the fluid stresses; σ'_{ij} the deviatoric stress; p the hydrostatic pressure; v_i the velocity of a fluid particle; V_{ij} the velocity strains, $\bar{T}_i$ the prescribed tractions on a boundary segment S_t; $\bar{v}_i$ the prescribed velocities at S_v; and $(\ )_{,j}$ denotes the partial differentiation w.r.t. x_j. The well-known ('Navier–Stokes') field equations are:

$$\text{(Incompressibility):} \quad v_{i,i} = 0 \text{ in V} \tag{1}$$

$$\text{(Momentum balance):} \quad \sigma_{ij,j} + \rho\bar{F}_i = \rho\frac{\partial v_i}{\partial t} + \rho v_{i,j}v_j \tag{2a}$$

$$\sigma_{ij} = \sigma_{ji} \tag{2b}$$

$$\text{(Compatibility):} \quad V_{ij} = V_{ji} = v_{(i,j)} \equiv \tfrac{1}{2}(v_{i,j} + v_{j,i}) \tag{3}$$

$$\text{(Constitutive law)*:} \quad \sigma_{ij} = \partial A/\partial V_{ij} \tag{4}$$

where

$$A(V_{lm}, p) := -pV_{kk} + \mu V_{lm}V_{lm} \tag{5}$$

μ is the coefficient of viscosity, and thus

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu V_{ij} \tag{6}$$

$$\text{(Traction b.c):} \quad \sigma_{ij}n_j \equiv T_i = \bar{T}_i \text{ at } S_t \tag{7}$$

$$\text{(Velocity b.c.):} \quad v_i = \bar{v}_i \text{ at } S_v \tag{8}$$

where n_j are components of a unit outward normal to S. We further note that:

$$\sigma_{ij} = -p\,\delta_{ij} + \sigma'_{ij} \tag{9}$$

Thus,

$$\sigma'_{ij} = 2\mu V_{ij} \tag{10}$$

It is well-known that for all compatible velocity fields v_i, which satisfy equation (8) *a priori*,

* Here A is the stress-working density per unit volume.

but not equation (1) *a priori*, the variational statement,

$$\delta P + \delta N = 0 \tag{11}$$

where

$$P(p, v_i) = \int_V [A(p, v_i) - \rho \bar{F}_i v_i]\,\mathrm{d}V - \int_{S_t} \bar{T}_i v_i\,\mathrm{d}s \tag{12}$$

and

$$\delta N = \int_V \rho\left[\frac{\partial v_i}{\partial t} + v_{i,j} v_j\right] \delta v_i\,\mathrm{d}V \tag{13}$$

for all arbitrary variations δp, and admissible variations δv_i[i.e. $\delta V_{ij} = \delta v_{(i,j)}$ in V and $\delta v_i = 0$ at S_v], leads to the Euler–Lagrange equations which are the above equations (1), (2) and (7), respectively. In an approximate solution* based on equation (11), if the same basis functions are chosen for v_i as well as for δv_i, the method is referred to as a standard Galerkin weighted residual method. On the other hand, if the basis functions chosen for δv_i are different from those for v_i, the method is in general referred to as the Petrov–Galerkin weighted residual method.

Suppose now that the constraints, equations (3) and (8) are relaxed through Lagrange multipliers σ_{ij} and T_i respectively. Thus a general variational statement which leads as its Euler–Lagrange equations, to equations (1)–(3), (7) and (8), respectively, is given by:†

$$\delta G + \delta N = 0 \tag{14}$$

where,

$$G(p, V_{ij}, v_i, \sigma'_{ij}) = \int_V \{-pV_{kk} + \mu V_{lm} V_{lm} - \sigma_{ij}\langle V_{ij} - v_{(i,j)}\rangle - \rho \bar{F}_i v_i\}\,\mathrm{d}V$$
$$- \int_{S_t} \bar{T}_i v_i\,\mathrm{d}S + \int_{S_v} \sigma_{ij} n_j (\bar{v}_i - v_i)\,\mathrm{d}S \tag{15}$$

and δN is as in equation (13). In equation (15) it is understood that $\sigma_{ij} = -p\,\delta_{ij} + \sigma'_{ij}$.

We now consider the Legendre contact transformation to express the stress working density of the fluid in terms of stresses. Thus, let

$$-B(\sigma_{km}) = A(p, V_{km}) - \sigma_{km} V_{km} \tag{16}$$

Such that,

$$V_{ij} = \partial B / \partial \sigma_{ij} \tag{17}$$

Using equations (5), (9), and (10) in (16), we find:

$$B(\sigma_{km}) = \frac{1}{4\mu}\,\sigma'_{km}\sigma'_{km}. \tag{18}$$

Thus, for an incompressible fluid, the complementary stress-working density B depends on the stress-deviator alone.

Now, we eliminate V_{km} as a variable from equation (15) by using equation (16), and thus

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* In a finite element solution, since A is quadratic in $v_{(i,j)}$, it is seen that v_i should be C^0 continuous in each element as well as at the element boundaries.

† This procedure is formally analogous to that used in deriving the so-called Hu–Washizu principle in solid mechanics. See, for instance, Reference 32.

obtain:

$$R(p, v_i, \sigma'_{ij}) = \int_V \{-B(\sigma_{km}) + \sigma_{ij} v_{i,j} - \rho \bar{F}_i v_i\} \, dV - \int_{S_t} \bar{T}_i v_i \, dS + \int_{S_v} \sigma_{ij} n_i (\bar{v}_i - v_i) \, dS \qquad (19)$$

It can easily be verified that the variational statement,*

$$\delta R + \delta N = 0 \qquad (20)$$

with δN as in equation (13), leads to: (i) incompressibility equation (1) and momentum balance equation (2), (ii) compatibility, $\partial B/\partial \sigma_{km} = V_{km}$, (iii) traction b.c., equation (7), and (iv) velocity b.c., equation (8).

Now, we consider a discretization of the flow domain into 'finite elements' V_m, $m = 1, 2, \ldots, N$. Let the boundary of V_m be denoted by ∂V_m. It is seen that, in general, $\partial V_m = \rho_m + S_{tm} + S_{vm}$, where ρ_m is the interelement boundary, and S_{tm} and S_{vm} are those parts of ∂V_m where tractions and velocities, respectively, are prescribed. It is also seen that for those elements whose boundaries do not coincide with the external boundaries of the flow domain, one has, in general, that $\partial V_m = \rho_m$. In the finite element model, the field equations of (i) incompressibility and momentum balance, equations (1) and (2); (ii) compatibility, equation (3); (iii) the constitutive law, equation (17); and (iv) the boundary conditions equations (7), (8) should be satisfied for each element V_m. These will be referred to as the intraelement constraints. In addition, the interelement constraint conditions, (i) velocity compatibility, $v_i^+ = v_i^-$ at ρ_m and (ii) the traction reciprocity condition $(\sigma_{ij} n_j)^+ + (\sigma_{ij} n_j)^- = 0$ at ρ_m, where $(+)$ and $(-)$ arbitrarily denote the two sides of ρ_m, should be satisfied. Thus, in a finite element method based on equations (20), (19) and (13), one may choose v_i, σ'_{ij}, and p such that both the interelement velocity compatibility and traction reciprocity are satisfied *a priori* and allow the intraelement constraints to be satisfied *a posteriori* through the variational statement, equation (20). This can be done, for instance, by choosing each of the three fields, v_i, σ'_{ij}, and p to be C° continuous over each element in terms of their respective nodal values. Thus, in the final finite element system of equations, the nodal values of v_i, σ'_{ij} and p would remain as unknowns. This approach would be analogous to that in solid mechanics as typified, for instance, by the work of Olson[33] and Wunderlich.[34] However, as demonstrated for instance in Reference 35, in the case of the Hellinger–Reissner principle of solid mechanics, which is formally analogous to the present equation (20), if in the finite-element counterpart of equation (20) (wherein all the integrals are simply replaced by the sum of the respective integrals over each of the elements) the velocity field is interelement compatible *a priori*, the traction reciprocity is satisfied *a posteriori* through the resulting finite-element variational principle. In this case, the stress field σ'_{ij} and p can be arbitrary in each element, i.e. they are neither subject to intraelement nor interelement constraints. Here we consider only a C° continuous velocity field and further assume, without loss of generality that the velocity field also satisfies, *a priori*, the velocity condition at S_{vm}, equation (8), and assume arbitrary σ'_{ij}, p fields which are not subject to either intraelement or interelement constraints *a priori*. Then, the finite element counterpart of equation (20) can be stated as:

$$\delta FR + \delta FN = 0 \qquad (21)$$

* It is seen that this variational statement is formally analogous to the well-known Hellinger–Reissner principle of solid mechanics.[32]

where

$$FR(p, v_i, \sigma'_{ij}) = \sum_m \left\{ \int_{V_m} [-B(\sigma_{km}) + \sigma_{ij} v_{i,j} - \rho \bar{F}_i v_i] \, dV - \int_{S_{tm}} \bar{T}_i v_i \, dS \right\} \tag{22}$$

and

$$\delta FN = \sum_m \int_{V_m} \rho \left[\frac{\partial v_i}{\partial t} + v_{i,j} v_j \right] \delta v_i \, dV \tag{23}$$

Then it can easily be verified that for all admissible (C^0 continuous) δv_i and arbitrary $\delta\sigma'_{ij}$ and δp, equation (21) leads to the Euler–Lagrange equations (1)–(3), (7) and the interelement traction reciprocity condition. Further, since σ'_{ij} is arbitrary (in the sense of not being subject *a priori* to either intraelement or to interelement constraints) in each element the unknown parameters in σ'_{ij} can be solved for at the element level and expressed in terms of the nodal velocities, from the algebraic equations resulting from the stationary condition $\delta(FR) + \delta(FN) = 0$ for all $\delta\sigma'_{ij}$. Thus, the final finite element equations would involve only the nodal values of $\mathbf{u}$ and p as unknowns. However, this approach would then become analogous to the standard 'primitive variable' 'velocity-pressure' mixed method. This situation is analogous to that in solid mechanics. For in solid mechanics, in connection with mixed methods based on Hellinger–Reissner principle, the so-called 'limitation principle' has been established by Fraeijs de Veubeke,[36] which states that, if no constraints are imposed on the assumed stress distribution and the stress parameters are eliminated at the element level, the Hellinger–Reissner principle will yield the same stiffness matrix as that by the assumed displacement method. Thus, in solid mechanics, in a mixed method based on assumed stresses which are eliminated at the element level, it is generally necessary to impose constraints on the chosen stresses in order to yield a discrete formulation that is different from that based on assumed displacements. In linear solid mechanics, if in the Hellinger–Reissner principle the stress field is subject *a priori* to the intraelement momentum balance constraint, but not the interelement traction reciprocity condition (which still makes it possible to eliminate the stresses at the element level), one obtains the so-called hybrid-stress formulation[35,37] which is known to possess several advantages[37] over the standard displacement method. Likewise, in the case of slow-creeping motion of a fluid ('Stokes flow'), the deviatoric stress field σ'_{ij} and p may be subject to the intraelement moment balance condition $(\sigma'_{ij,j} - p_{,i} + \rho \bar{F}_i = 0)$ *a priori*, but not to the interelement traction reciprocity condition, and obtain a so-called hybrid formulation as shown by one of the authors.[28,29] As demonstrated in References 28 and 29, in the hybrid method for Stokes flow, the final finite element equations would involve only the constant term in the pressure field in each element as an unknown in addition to nodal velocities. Now we turn to the present case of Navier–Stokes equations wherein the momentum balance condition in terms of σ'_{ij} and p involves the non-linear acceleration term in terms of v_i as in equation (2a). Owing to this assumption of a stress field (σ'_{ij}, p) to satisfy the momentum balance condition, equation (2a), *a priori* is a difficult, if not an impossible task. Consider the analogous problem in solid mechanics—for instance, that of non-linear (large deformation) elastodynamics. Here, owing to the Lagrangian description of motion, the inertia term is linear; the non-linearity is due to large deformation. The momentum balance condition, for instance, in terms of the 2nd Piola–Kirchhoff stress tensor is non-linear[38] in the sense that it involves coupling between the dependent variables, stress and displacement. Even for elasto-static large deformation problems, the momentum balance condition in its rate form (governing the stress-rate at time t) is of the type.[38]

$$\dot{S}_{ij,j} = -[(\tau_{kj}\dot{u}_{i,k})_{,j} + \rho_t \dot{\bar{F}}_i]$$

where $\dot{S}_{ij}$ is the rate of 2nd Piola–Kirchhoff stress, $\dot{u}_i$ is the rate of deformation, τ_{ij} are initial stresses, and $(\ )_{,j}$ denotes the differentiation w.r.t Cartesian co-ordinates in the solid body in its configuration at time t. Mixed variational principles for the rate problem, involving $\dot{S}_{ij}$ and $\dot{u}_i$ can be derived.[39,40] However, in their application, if $\dot{S}_{ij}$ is completely unconstrained and the parameters in $\dot{S}_{ij}$ are solved for at the element level, in terms of the element-nodal values of $\dot{u}_i$, the resulting finite element method becomes equivalent to the standard displacement-rate based method, in view of the earlier stated 'limitation principle' of Fraeijs de Veubeke.[36] Several ways of implementing a mixed method that is not subject to a limitation principle, by constraining $\dot{S}_{ij}$ either to satisfy the 'full' momentum balance condition as above or to satisfy only the homogeneous part (i.e. $\dot{S}_{ij,j}=0$), have been discussed in References 38–40. Several applications have been made of one of these types of mixed methods, wherein $\dot{S}_{ij}$ is constrained to satisfy only the homogeneous part of the momentum balance conditions, to large displacement, large rotation problems; see, for instance, References 40–43, wherein the thus obtained numerical results have been shown to be superior as compared with those based on standard displacement methods using similar spatial discritization. Apart from the validity of this type of mixed method as borne out by the several numerical results in References 41–43, a theoretical justification has also been provided by one of the authors.[44] Turning to the present application to Navier–Stokes equations, in order to generate a mixed method that is not subject to the 'limitation principle' of Fraeijs de Veubeke,[36] we constrain the stress field σ_{ij} so as to satisfy only the 'homogeneous' part of the momentum balance condition, namely

$$\sigma_{ij,j} + \rho\bar{F}_i = 0 = \sigma'_{ij,j} - p_{,i} + \rho\bar{F}_i = 0 \quad \text{in } V_m \tag{24}$$

It is seen that the above type of 'homogeneous' constraint on fluid stresses σ_{ij} (sans inertial terms) is entirely analogous, conceptually, to the homogeneous constraint imposed on stress rates $\dot{S}_{ij}$ in solid mechanics, as elaborated above. Thus, the justification for constraining σ_{ij} *a priori* as in equation (24) is entirely analogous to that in non-linear solid mechanics and will be indicated later in this paper in the context of the developed mixed finite element method for the full Navier–Stokes' equations. Now, in view of the constraint, equation (24), we reduce equation (22) to:

$$FM(p, v_i, \sigma'_{ij}) = \sum_m \left\{ \int_{V_m} -B(\sigma_{km})\,\mathrm{d}V + \int_{\partial V_m} n_j(\sigma'_{ij} - p\,\delta_{ij})v_i\,\mathrm{d}S - \int_{S_{tm}} \bar{T}_i v_i\,\mathrm{d}S \right\} \tag{25}$$

Thus, the present mixed finite element method follows from the variational statement,

$$\delta FM + \delta FN = 0 \tag{26}$$

with (FM) as in equation (25) and $\delta(FN)$ as in equation (23).

In order to understand the justification of the present mixed method based on the constraint equation (24), it is instructive to study the meaning of the discrete equations that one obtains from equation (25). We first note that:

$$\begin{aligned} \delta FM + \delta FN = 0 = \sum \Bigg\{ & \int_{V_m} -\frac{\partial B}{\partial \sigma_{km}}\,\delta\sigma_{km}\,\mathrm{d}V + \int_{\partial V_m} n_j(\sigma'_{ij} - p\,\delta_{ij})\,\delta v_i\,\mathrm{d}S \\ & + \int_{\partial V_m} n_j(\delta\sigma'_{ij} - \delta p\,\delta_{ij})v_i\,\mathrm{d}S - \int_{S_{tm}} \bar{T}_i\,\delta v_i\,\mathrm{d}S + \int_{V_m} \rho\left[\frac{\partial v_i}{\partial t} + v_{i,j}v_j\right]\delta v_i\,\mathrm{d}V \Bigg\} \end{aligned} \tag{27}$$

Since $\delta\sigma_{ij,j}=0$, we may write equation (27) as:

$$\delta FM+\delta FN=0=\sum_m\left\{\int_{V_m}\left(-\frac{\partial B}{\partial\sigma_{ij}}+v_{(i,j)}\right)\delta\sigma_{ij}\,\mathrm{d}V+\int_{\partial V_m}n_j(\sigma'_{ij}-p\,\delta_{ij})\,\delta v_i\,\mathrm{d}S\right.$$
$$\left.-\int_{S_{tm}}\bar{T}_i\,\delta v_i\,\mathrm{d}S+\int_{V_m}\rho\left[\frac{\partial v_i}{\partial t}+v_{i,j}v_j\right]\delta v_i\,\mathrm{d}V\right\} \tag{28}$$

Now, assuming, without loss of generality, that $\bar{F}_i=0$, let the stress field that satisfies the constraint equation (24) be chosen as:

$$\sigma'_{ij}=\sigma'_{ijn}\beta_n \qquad (n=1,\ldots,N_\beta) \tag{29}$$
$$p=\alpha_1+A_m\alpha_m \qquad (m=2,\ldots,N_\alpha) \tag{30a}$$

where σ'_{ijn} and A_m are functions of x_i, and β_n and α_m are undetermined coefficients. The constraint equation (24), in general, implies that

$$A_m\alpha_m=D_n\beta_n \qquad (n=1\ldots N_\beta;\quad m=2\ldots N_\alpha), \tag{30b}^*$$

whereas α_1 remains undetermined in each element. Note also that *not all* the N_β parameters need, in general, to enter the representation for p.

Finally, let the velocity field that is C° continuous over each element be such that:

$$v_k=B_{kr}q_r \qquad (r=1,\ldots,N_q) \tag{31}$$

In the above, N_β is the number of undetermined parameters in σ'_{ij}, α_1 is the undetermined 'constant' term in p in each element, and N_q is the number of nodal velocity degrees of freedom.

The discrete form of equations can now be written by using equations (29)–(31) in equation (28) as:

$$0=\sum_{e=1}^{N}\left\{\int_{V_e}\left[\left\langle-\frac{1}{2\mu}\sigma'_{ijn}\beta_n+\tfrac{1}{2}(B_{ir,j}+B_{jr,i})q_r\right\rangle\sigma'_{ijl}\,\delta\beta_l-B_{ir,i}q_r(\delta\alpha_1+D_n\,\delta\beta_n)\right]\mathrm{d}V\right.$$
$$+\int_{\partial V_e}n_j[\sigma'_{ijn}\beta_n-\delta_{ij}(\alpha_1+D_n\beta_n)]B_{ir}\,\delta q_r\,\mathrm{d}s$$
$$\left.-\int_{S_{tm}}\bar{T}_iB_{ir}\,\delta q_r\,\mathrm{d}S+\int_{V_e}\rho\left[B_{ik}\frac{\partial q_k}{\partial t}+B_{ir,j}q_rB_{js}q_s\right]B_{il}\,\delta q_l\,\mathrm{d}V\right\} \tag{32}$$

Since the parameters β_i and α_1 are arbitrary for each element (in the sense of being not subject *a priori* to any interelement constraint), the first two terms can be seen to lead to the discretized physical law that the strains corresponding to the assumed stresses are equal to the assumed velocity strains and that the assumed velocity field obeys the incompressibility constraint, both in an integral average sense.

In order to understand the physical implication of the last two terms in equation (32) we first note that q_k are subject to nodal connectivity; and that B_{kr} have only a local support in the usual finite element sense. Thus, consider a 'patch' with several elements meeting at a node. Thus a virtual velocity at this node affects only the surrounding elements. Consider a virtual velocity δq_k at this node. Let the number of elements in the 'patch' surrounding this

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* This relation is given explicitly for some specific element formulations in the Appendix I.

node be N_p. Thus equation (32) implies that:

$$\sum_{\substack{\text{ele}\\e=1}}^{N_p}\left\{\left\langle\int_{\partial V_e} n_j[\sigma'_{ijn}\beta_n-\delta_{ij}(\alpha_1+D_n\beta_n)]B_{ir}-\int_{S_{te}}\bar{T}_iB_{ir}+\int_{V_e}\rho\left[B_{ir}B_{ik}\frac{\partial q_k}{\partial t}+B_{ik,j}B_{jS}q_kq_SB_{ir}\right]\right\rangle\delta q_r\right\}=0 \tag{33}$$

Thus, for the above patch, owing to an arbitrary virtual velocity at the 'centre node' of the patch, equation (33) implies the balance, in an integral average sense over the patch, between: (i) the rate of work of unequilibrated tractions at the interelement boundaries within the patch; (ii) rate of work of external tractions on the patch, and (iii) the rate of work of the convective acceleration forces.

Recall that in the present formulation, σ'_{ij} and p are only constrained within each element to satisfy the 'homogeneous' momentum balance condition (sans inertial terms) *a priori* but not the interelement traction reciprocity condition. Thus, even though σ_{ij} and p are subject to the intraelement constraint condition of equation (24), it is the work of the unequilibrated tractions that produces the convective acceleration, as evident from equation (33). Thus, equation (33) is the physical justification for the present mixed method based on the constraint equation (24). This is entirely analogous to the case of non-linear solid mechanics[44] discussed earlier.

Comparisons of the discrete (finite element) equations that arise out of the present mixed method, i.e. equations (32) and (33), with those that would arise from the standard velocity–pressure formulation are given in Appendix II, which is intended to further clarify the present method and its distinction from the standard velocity–pressure formulation.

We now proceed with constructing the finite element equations for the present mixed method. In doing so, we write (FM) of equation (25), using equations (29)–(31), as:

$$\begin{aligned} FM=\sum_{\substack{\text{ele}\\e=1}}^{N}&\left\{\int_{V_e}-\frac{1}{4\mu}\sigma'_{ijn}\sigma'_{ijm}\beta_n\beta_m\,\mathrm{d}V\right.\\ &\left.+\int_{\partial V_e}n_j[\sigma'_{ijn}\beta_n-\delta_{ij}(\alpha_1+D_n\beta_n)]B_{ir}q_r\,\mathrm{d}s-\int_{S_{te}}\bar{T}_iB_{ir}q_r\,\mathrm{d}s\right\}\\ =\sum_{e=1}^{N}&\{-\tfrac{1}{2}H_{nm}\beta_n\beta_m+G_{nr}\beta_nq_r-\alpha_1S_rq_r-Q_rq_r\} \end{aligned} \tag{34}$$

Thus, the meanings of H_{nm}, G_{nr}, S_r, and Q_r are apparent from equation (34). Since β'_m are independent for each element, and (δFN) of equation (23) is independent of β'_m, equation (26) leads to an element-level equation,

$$\delta(FM)=\frac{\partial(FM)}{\partial\beta_m}\delta\beta_m=0\Rightarrow H_{nm}\beta_n=G_{mr}q_r \tag{35a}$$

i.e.

$$\beta_n=H^{-1}_{nm}G_{mr}q_r\quad\text{in } V_e \tag{35b}$$

We note here that equation (35b) expresses the element stress parameters β_n in terms of the element nodal velocities that are yet to be solved for.

Using equation (35) we write

$$FM=\sum_e(\tfrac{1}{2}k_{rS}q_rq_S-\alpha_1S_rq_r-Q_rq_r) \tag{36}$$

where

$$k_{rs}=G_{nr}H^{-1}_{nm}G_{ms} \tag{37}$$

Likewise,

$$\delta(FN)=\sum_e\left\{\int_{V_e}\rho\left[B_{ir}B_{is}\frac{\partial q_s}{\partial t}+B_{it,j}B_{js}B_{ir}q_tq_s\right]\delta q_r\,\mathrm{d}V\right\}$$
$$=\sum_e\left(M_{rs}\frac{\partial q_s}{\partial t}+C_{rst}q_sq_t\right)\delta q_r \tag{38}$$

Thus, $\delta(FM)+\delta(FN)=0$ leads to:

$$0=\sum_e\left\{\left(M_{rs}\frac{\partial q_s}{\partial t}+k_{rs}q_s+C_{rst}q_sq_t-\alpha_1S_r-Q_r\right)\delta q_r-(S_rq_r)\,\delta\alpha_1\right\} \tag{39}$$

Equation (39) leads to the global finite element system of equations,

$$\mathbf{M}\frac{\partial\mathbf{q}^*}{\partial t}+[\mathbf{K}_{\text{linear}}+\mathbf{K}_{\text{conv.}}(\mathbf{q}^*)]\mathbf{q}^*-\mathbf{S}^*\boldsymbol{\alpha}=\mathbf{Q}^* \tag{40a}$$

$$-\mathbf{S}^{*T}\mathbf{q}^*=0 \tag{40b}$$

where $\mathbf{K}_{\text{linear}}$ is the linear term independent of $\mathbf{q}^*$ and $\mathbf{K}_{\text{conv.}}$ arises from the convective acceleration and depends on $\mathbf{q}^*$. Note that the dimension of the vector $\boldsymbol{\alpha}$ is equal to the number of elements (say N), and if the number of nodal velocities is M, then $\mathbf{S}^*$ is an $(M\times N)$ matrix.

It is worthwhile to point out that the element nodal velocities (a subset of the global vector $\mathbf{q}^*$) are solved from the full non-linear equations including convection terms as in equation (40). Thus the element stresses as solved from equation (35b) would reflect the influence of fluid acceleration.

Before proceeding with a discussion of certain fundamental concepts of stability of the above numerical scheme, and the techniques employed presently to solve the non-linear system of equations (40), we briefly remark on some salient features of the present approach.

Remark 1

For a given finite-element mesh, the system of equations (40a,b) is larger than that arises in the currently popularized reduced-integration-penalty methods[9–11] only by the *number of elements* in the mesh, N.

Remark 2

In the present formulation, the hydrostatic pressure field in each element can be an arbitrary polynomial. However, it is only the constant term of this polynomial that becomes a solution variable in the finite element system of equations. Thus, in contrast to the velocity-pressure formulations,[1–5] wherein all the nodal pressures are global unknowns, in the present method only the 'constant' term of the pressure field in each element is a global unknown. Thus, the present system of equations will, in general, be smaller than that in References 1–5.

Remark 3

We have labelled the present method also as a mixed method. However, the present method involves the assumption of the deviatoric stress field, the pressure field, and the velocity field in each element. The earlier 'primitive-variable' finite element schemes based on velocity–pressure formulations have also been called mixed methods. In this earlier category of mixed methods, pressure acts as a direct Lagrange multiplier to enforce the incompressibility constraint. It is to this second category of mixed methods (based on velocity–pressure) that the so-called 'equivalence theorem' has been argued[22] to be applicable. This equivalence theorem suggests that to each mixed finite element of the above second category, with continuous velocities and discontinuous pressure fields, there corresponds an element and a reduced-integration scheme in the penalty function formulation.[22] However, the present mixed method is not in general 'equivalent' to any reduced-integration-penalty method.

Remark 4

No reduced or selective-reduced numerical integrations are employed in the present method. All integrations are performed with necessary order quadrature rules. Although this may not be an advantage over selective-reduced integration schemes, it does eliminate the degree of arbitrariness and numerical experimentation associated with the later schemes.

Remark 5

In the present method, no 'upwinding' techniques are used; the convective acceleration term is treated via the standard Galerkin technique.

We now comment on certain aspects of stability of the present scheme and the techniques employed presently in solving the non-linear system of equations (40).

STABILITY OF THE NUMERICAL SCHEME

Here we do not embark on a detailed discussion of the LBB conditions governing this type of mixed method; however, for the associated linear problems, the two LBB conditions that govern the stability and convergence of the scheme have been proved[29] to be:

$$\operatorname*{Sup}_{(\sigma'_{ij},\, p')\in\sigma_h} \frac{\sum_{k=1}^{N} \int_{\partial\Omega_k} v_i(\sigma'_{ij} - p'\,\delta_{ij}) n_j}{\|(\sigma'_{ij}, p')\|_{H_0(\Omega)}} \geqslant \beta^* \|\mathbf{v}\|_{(H_0^{1/2}(\Gamma))^n}\ \forall \mathbf{v} \in V_{h0} \tag{41}$$

$$\operatorname*{Sup}_{\mathbf{v}\in V_h} \frac{\sum_{k=1}^{N} p_k \int_{\partial\Omega_k} v_i n_i \, ds}{\|\mathbf{v}\|_{(H_0^{1/2}(\Gamma))^n}} \geqslant \gamma^* \|p_c\|_P\ \forall p_c \in P \tag{42}$$

If equations (41) and (42) are met, then the finite element problem has a unique solution. However, if β^* and γ^* do not depend on the mesh parameter h then convergence can be established. A detailed discussion of the satisfaction of the above conditions for 2-dimensional elements, which are used in the present work, has been presented in Reference 29. Here we restrict ourselves to present certain new ideas that have been generated since Reference 29 was written towards choosing σ'_{ij} and $\mathbf{p}'$ in each element such that satisfaction of equation (41) is assured *a priori.*

We first note the form of the linear 'stiffness' of each element, given in equation (37), to be:

$$\mathbf{k} = \mathbf{G}^{\mathrm{T}}\mathbf{H}^{-1}\mathbf{G} \tag{43}$$

where,

$$\tfrac{1}{2}\boldsymbol{\beta}^{\mathrm{T}}\mathbf{H}\boldsymbol{\beta}=\int_{V_e}\frac{1}{4\mu}\sigma'_{ijn}\sigma'_{ijm}\beta_n\beta_m\,\mathrm{d}V \tag{44}$$

and

$$\begin{aligned}\beta^{\mathrm{T}}\mathbf{G}q&=\int_{V_e}(\sigma'_{ij}-p^*\,\delta_{ij})v_{(i,j)}\,\mathrm{d}V\\&=\frac{1}{2}\int_{V_e}(\sigma'_{ijn}-D_n\,\delta_{ij})\beta_n(B_{ir,j}+B_{jr,i})q_r\,\mathrm{d}V\end{aligned} \tag{45}$$

In equation (45), $p^*=p-\alpha_1$.

As mentioned earlier, the number of undetermined parameters in σ'_{ij} (and hence in the non-constant part of p, i.e. p^*) is N_β; the number of generalized nodal velocities is N_q; and the number of rigid body modes for each element is N_r (i.e. $N_r=3$ for planar elements, and 6 for 3-D elements). It is then noted that:

(i) From equation (44) it is seen that since $B(\sigma_{ij})=(1/4\mu)\sigma'_{ij}\sigma'_{ij}$ is positive definite for all chosen σ'_{ij}, the $\mathbf{H}$ matrix is always positive definite and has the rank N_β.

(ii) Even though equation (43) appears to indicate the need for inverting $\mathbf{H}$ explicitly, in reality, since $\mathbf{H}^{-1}\mathbf{G}$ appears in equation (43), this term can be evaluated directly from equation (35a) by an equation solver with multiple 'right-hand sides'. This is in fact much less expensive than explicitly finding $\mathbf{H}^{-1}$.

(iii) A 'good' linear 'stiffness' matrix for the element should involve all the rigid body modes of the element. Thus the rank of the element $\mathbf{k}$ should be N_q-N_r.

(iv) The matrix $\mathbf{G}$ is of order $(N_\beta\times N_q)$. From equation (45) it is seen that since $V_{ij}=v_{(i,j)}$ is zero for N_r rigid modes, the rank of $\mathbf{G}$ is, at best, the minimum of $(N_\beta;N_q-N_r)$.

(v) In view of (iv) it is seen that the rank of $\mathbf{k}$ of equation (43), is, at best, the minimum of $(N_\beta;N_q-N_r)$. Thus, in view of the requirement (iii), it is seen that one must have the criterion, $N_\beta\geq N_q-N_r$. For simplicity, one may choose $N_\beta=N_q-N_r$.

(vi) The central problem becomes one of assuring by a careful choice of σ_{ij} in V_e, for a given v_i, that the rank of $\mathbf{G}$ is N_q-N_r.

Since V_{ij} in equation (45) involves only (N_q-N_r) parameters, in order to assure the rank of $\mathbf{G}$ to be N_q-N_r, we must choose $N_\beta(\geq N_q-N_r)$ equilibrated stress modes of $(\sigma'_{ij}-p^*\,\delta_{ij})$ in each element such that

$$\int_{V_e}(\sigma'_{ij}-p^*\,\delta_{ij})V_{ij}\,\mathrm{d}V>0;\qquad V_{ij}\neq 0 \tag{46}$$

for each of the components of $(\sigma'_{ij}-p^*\,\delta_{ij})$. This condition is seen to be necessary for the satisfaction of equation (41).

Even though several elements were developed to satisfy the above rank conditions, some of which are illustrated in Appendix I, we discuss here the simplest element and the one used to generate the solutions reported in Part II of this paper—the 'four noded' isoparametric element. The degenerate case of this element, namely the square, is first treated.

It has been shown[31] that concepts of group theory are helpful—and essential—in choosing σ_{ij} to satisfy the rank condition and equation (46). For the purposes of our present discussion we wish not to repeat these group theoretical arguments[31] but indicate only the essential ideas.

The most direct attack on the above 'rank problem' (also, physically, the problem of 'mechanism modes'[31] is to choose as a stress interpolation any complete equilibrated polynomial field having at least as many degrees of freedom as the V_{ij}, to form the $\mathbf{G}$ matrix, and then to compute the rank of $\mathbf{G}$ by a procedure such as Gaussian elimination. If the rank proves too small, the next highest order stresses must be added to the interpolation. When the rank of $\mathbf{G}$ equals $(N_q - N_r)$, this elimination process should reveal which stress degrees of freedom are redundant. In practice, especially for 3-dimensional elements, this straightforward method is cumbersome since the matrix $\mathbf{G}$ will be both large and relatively dense. Moreover, it is a non-trivial matter to eliminate redundant stresses, while at the same time preserving the invariance of the stress-interpolants.

The requirement of this invariance alone suggests the relevance of group representation theory (see for instance Reference 45) to this problem. In the present discussion we illustrate the use of such a theory and show that it leads to a sparse, quasidiagonal $\mathbf{G}$ matrix from which we can easily determine the rank of $\mathbf{G}$.

Consider a square element with cartesian coordinates (x, y) located at the centroid. The symmetry group G of the square, consisting of rotations and reflections, has the following representation:[45]

$$C_1\colon \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}; \quad C_2\colon \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}; \quad C_3\colon \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$
$$C_4\colon \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}; \quad C_5\colon \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix} \tag{47}$$

Class C_1 transforms (x, y) onto itself, C_2 transforms (x, y) to $(-x, -y)$, etc. As shown by Burnside[45] the above group has 5 irreducible representations, which we label here as $\Gamma_1, \Gamma_2, \ldots, \Gamma_5$.

For a 4-noded square, the velocity representation is:

$$\mathbf{v} = (1, x, y, xy)X + (1, x, y, xy)Y \tag{48}$$

In equation (48) the simple dyadic notation has been used: X, Y denote the direction of base vectors and x, etc. represent the polynomial components. The velocity strains corresponding to equation (48) are:

$$\mathbf{V} = (0, 1, 0, y)XX + (0, 0, 1, x)YY + (0, 1, x, y)XY \tag{49}$$

Using the group theoretical concepts[45] as illustrated in detail in Reference 31, we obtain the strain decomposition into irreducible subspaces, $\Gamma_1 \ldots \Gamma_5$, respectively, as follows:

$$\Gamma_1\colon \mathbf{V}_1 = \mu_1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}; \quad \Gamma_2\colon \mathbf{V}_2 = \mu_2\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix};$$
$$\Gamma_3\colon \mathbf{V}_3 = \mu_3\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}; \quad \Gamma_5\colon \mathbf{V}_5 = \mu_4\begin{bmatrix}y & x/2\\x/2 & 0\end{bmatrix} + \mu_5\begin{bmatrix}0 & y/2\\y/2 & x\end{bmatrix} \tag{50}$$

The strains $\mathbf{V}$ in equation (50) may be viewed as 'natural strains'.

We consider only a constant pressure $(p = \alpha_1)$ in each element. Thus $p^* = 0$. The equilibrated deviatoric stress of linear variation in x, y can be assumed as:

$$\boldsymbol{\sigma}' = [\alpha_1 + \alpha_2 x + \alpha_3 y]XX + [\alpha_4 + \alpha_5 x + \alpha_6 y]YY + [\alpha_7 - \alpha_6 x - \alpha_2 y]XY \tag{51}$$

The irreducible representation of the above stress field (or the 'natural stress' modes) is:

$$\Gamma_1\colon \boldsymbol{\sigma}_1 = \beta_1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}; \quad \Gamma_2\colon \boldsymbol{\sigma}_2 = \beta_2\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix};$$
$$\Gamma_3\colon \boldsymbol{\sigma}_3 = \beta_3\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}; \quad \Gamma_5\colon \sigma_5^{(1)} = \beta_4\begin{bmatrix}0 & x\\x & -y\end{bmatrix} + \beta_5\begin{bmatrix}-x & y\\y & 0\end{bmatrix}$$
$$\boldsymbol{\sigma}_5^{(2)} = \beta_6\begin{bmatrix}y & 0\\0 & 0\end{bmatrix} + \beta_7\begin{bmatrix}0 & 0\\0 & x\end{bmatrix} \tag{52}$$

In equation (52) it implies that $\sigma_5^{(1)}$ and $\sigma_5^{(2)}$ are two linearly independent bases in the irreducible representation Γ_5.

The advantage of representing $\mathbf{V}$ and $\boldsymbol{\sigma}'$ through irreducible representations as in equations (50) and (52), respectively, is now apparent. We now compute the term

$$\boldsymbol{\beta}\mathbf{G}\boldsymbol{\mu} = \int_{V_e} \sigma'_{ij} V_{ij} \, \mathrm{d}V \tag{53}$$

using the representation of equations (50) and (52). The matrix $\mathbf{G}$ can easily be found to be (for a square element $-1 \leq x, y \leq +1$):

$$\begin{array}{c|c|c|c|cc|}
 & \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 & \mathbf{V}_5 & \\ \hline
\boldsymbol{\sigma}_1 & 8 & & & & \\ \hline
\boldsymbol{\sigma}_2 & & 8 & & & \\ \hline
\boldsymbol{\sigma}_3 & & & 8 & & \\ \hline
\boldsymbol{\sigma}_5^{(1)} & & & & 4/3 & 0 \\
 & & & & 0 & 4/3 \\ \hline
\boldsymbol{\sigma}_5^{(2)} & & & & 4/3 & 0 \\
 & & & & 0 & 4/3 \\ \hline
\end{array} \equiv \mathbf{G} \tag{54}$$

Thus, in terms of the natural strain and stress decompositions, the $\mathbf{G}$ matrix becomes quasi-diagonal in nature. From an observation of $\mathbf{G}$ in equation (54), in order to maintain the rank of $\mathbf{G}$ to be 5 [$N_q = 8$, $N_r = 3$, $N_q - N_r = 5$], it is necessary to include at least 5 ($N_\beta = 5$, $N_\beta = N_q - N_r$) stress-modes, $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$, and ($\boldsymbol{\sigma}_5^{(1)}$ *or* $\boldsymbol{\sigma}_5^{(2)}$). This 5 parameter stress field would be of *least order*, yet *stable* and *invariant*. However, a 7 parameter stress field containing *both* $\boldsymbol{\sigma}_5^{(1)}$ and $\boldsymbol{\sigma}_5^{(2)}$ which will lead to *complete* (up linear terms) in all the stresses σ'_{ij}, which is still *stable* and *invariant*, has been chosen in the present.

The above development, namely assurance of rank $\mathbf{G}$, had been for a square element. For an isoparametric element, namely

$$x = x(\xi, \eta); \qquad y = y(\xi, \eta); \qquad -1 \leq (\xi, \eta) \leq +1 \tag{55}$$

the stress field $\boldsymbol{\sigma}'$, in Cartesian co-ordinates, which satisfies equilibrium (in global co-ordinates, of course) would still be chosen as in equation (51) The velocity field can be chosen in isoparametric co-ordinates as,

$$\mathbf{v} = \mathbf{v}(\xi, \eta) \tag{56}$$

such that it is C^0 continuous. Of course, for a 4 noded element, $v(\xi, \eta)$ is bilinear in (ξ, η). It has been verified that the rank of $\mathbf{G}$ will still be 5 for even the most severely distorted elements when $\boldsymbol{\sigma}'$ and $\mathbf{v}$ are chosen as in equations (51) and (56), respectively, (of course, the evaluation of $\mathbf{G}$ from equation (53) now involves the usual Jacobian of isoparametric geometric mapping).

Even though the above 4-node element is the simplest to develop, and the one which is exclusively used to obtain the results presented in Part II of this paper, higher-order 2 and 3 dimensional elements which are *stable* can be (and have been) developed using the basic concepts presented above. Some of these elements are illutrated in Appendix I.

SOME DETAILS OF NUMERICAL SOLUTION OF EQUATION (40)

We first discuss the numerical aspects of solutions of *steady-flow* equations, namely equations (40a) and (40b) wherein the time-dependent term is omitted. For convenience, we rewrite these steady-state equations as:

$$\mathbf{F}(\mathbf{q}) = K(\mathbf{q})\mathbf{q} - \mathbf{f} = 0 \tag{57}$$

where $\mathbf{q}$ is the global nodal-velocity vector.

In the present series of computations, reported in Part II of this paper, to solve the non-linear, unsymmetric system of equations (57) we have used the so-called quasi-Newton method, with the so-called 'Broyden update'. The details of this algorithm are more or less analogous to those of Engelman, Strang, and Bathe.[26] Although these details are not repeated here, they are fully documented in Reference 46.

As for the numerical integration of the initial value problem for the unsteady flow as given in equations (40), we have used a one-step 'linearly implicit', predictor-corrector method as detailed for instance in Reference 9 among others. Other approaches such as the 'fractional step' method of Donea *et al.*[8] may also be used.

Thus, although no radically different schemes are being proposed for solving either the non-linear algebraic equations for steady-flow or for integrating the initial value problem of unsteady-flow, the present computations reported in Part II basically illustrate:

(i) the comparative merits of the spatial discretization using the present mixed 'deviatoric stress–pressure–velocity' formulations
(ii) the present treatment of incompressibility in comparison to that by reduced-integration-penalty methods
(iii) the use of the standard Galerkin scheme in conjunction with the present mixed formulation to treat the convection term, in contrast to the use of Petrov–Galerkin schemes and 'upwind' techniques
(iv) the use of necessary order quadrature rules (and consistent masses for the unsteady flow) in contrast to the use of selective reduced integration and/or diagonal mass representations.

CONCLUSIONS

A new mixed, assumed 'deviatoric stress–velocity–pressure', conventional Galerkin method has been presented. Its versatility, efficiency, and the comparison of its performance with that of the other methods, are indicated in Part II of this paper wherein solutions to a number of 2-dimensional flow problems are presented. The natural extension of the present method to solve 3-dimensional incompressible convection dominated flows is currently under way and will be reported on shortly.

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APPENDIX I: DETAILS OF SOME TWO-DIMENSIONAL ELEMENT FORMULATIONS

We first consider the four-noded element. In summary, the element formulation calls for:

(i) choice of σ'_{ij} and p to satisfy the constraint $(\sigma'_{ij,j} - p_{,i}) = 0$ (in the absence of prescribed body forces) in each element

(ii) choice of C° continuous velocity field. Within the four-noded element, we illustrate the details for two choices: (i) CPLSE4 element which has constant pressure, bilinear distribution of σ'_{ij}, and bilinear velocity; (ii) CPQSE4 which has constant pressure, quadratic σ'_{ij}, and bilinear velocities.

CPLSE4 element

(a) *Assumption of* σ'_{ij} and p. We start by assuming

$$p=\alpha_1 \tag{58}$$

$$\begin{aligned}\sigma'_{11}&=\beta_1+\beta_2x+\beta_3y+\beta_4xy\\ \sigma'_{12}&=\beta_5+\beta_6x+\beta_7y+\beta_8xy\\ \sigma_{22}&=\beta_9+\beta_{10}x+\beta_{11}y+\beta_{12}xy\end{aligned} \tag{59}$$

The constraint equations of 'homogeneous' momentum balance conditions, equation (24), for the present 2-D case are:

$$\begin{aligned}\frac{\partial\sigma'_{11}}{\partial x}+\frac{\partial\sigma'_{12}}{\partial y}-\frac{\partial p}{\partial x}&=0\\ \frac{\partial\sigma'_{12}}{\partial x}+\frac{\partial\sigma'_{22}}{\partial y}-\frac{\partial p}{\partial y}&=0\end{aligned} \tag{60}$$

Using (58) and (59) in (60), one obtains:

$$\begin{aligned}(\beta_2+\beta_4y)+(\beta_7+\beta_8x)&=0\\ (\beta_6+\beta_8y)+(\beta_{11}+\beta_{12}x)&=0\end{aligned} \tag{61}$$

which can be met by:

$$\beta_4=\beta_8=\beta_{12}=0;\qquad \beta_7=-\beta_2;\qquad \beta_{11}=-\beta_6 \tag{62}$$

Thus:

$$\begin{aligned}\sigma'_{11}&=\beta_1+\beta_2x+\beta_3y\\ \sigma'_{12}&=\beta_4+\beta_5x-\beta_2y\\ \sigma'_{22}&=\beta_6+\beta_7x-\beta_5y\\ p&=\alpha_1\end{aligned} \tag{63}$$

(b) *Assumption of* $\mathbf{v}$

$$v_x=\tfrac{1}{4}(1-\xi)(1-\eta)q_1+\tfrac{1}{4}(1+\xi)(1-\eta)q_2+\tfrac{1}{4}(1+\xi)(1+\eta)q_3+\tfrac{1}{4}(1-\xi)(1+\eta)q_4 \tag{64}$$

and a similar expression for v_y; and the geometry of the quadrilateral is described by:

$$x=\tfrac{1}{4}(1-\xi)(1-\eta)x_1+\tfrac{1}{4}(1+\xi)(1-\eta)x_2+\tfrac{1}{4}(1+\xi)(1+\eta)x_3+\tfrac{1}{4}(1-\xi)(1+\eta)x_4$$

CPQSE4 element

(a) *Assumption of* σ'_{ij}, p. Using procedures similar to above, we have:

$$\begin{aligned}p&=\alpha_1\\ \sigma'_{11}&=\beta_1+\beta_2x+\beta_3y+\beta_4xy+\beta_5x^2+\beta_6y^2\\ \sigma'_{12}&=\beta_7+\beta_8x-\beta_2y-2\beta_5xy+\beta_9x^2-\tfrac{1}{2}\beta_4y^2\\ \sigma'_{22}&=\beta_{10}+\beta_{11}x-\beta_8y-2\beta_9xy+\beta_{12}x^2+\beta_5y^2\end{aligned} \tag{65}$$

and the assumption of $\mathbf{v}$ is the same as in (64).

We now illustrate the formulation of an 8-noded quadrilateral element with quadratic pressure, quadratic deviatoric stresses, and quadratic velocities, herein designated as a QPQSE8.

QPQSE8 element

Assumption of σ'_{ij}, p. We start by assuming

$$\begin{aligned}
p &= \alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 xy+\alpha_5 x^2+\alpha_6 y^2 \\
\sigma'_{11} &= \beta_1+\beta_2 x+\beta_3 y+\beta_4 xy+\beta_5 x^2+\beta_6 y^2 \\
\sigma'_{12} &= \beta_7+\beta_8 x+\beta_9 y+\beta_{10} xy+\beta_{11} x^2+\beta_{12} y^2 \\
\sigma'_{22} &= \beta_{13}+\beta_{14} x+\beta_{15} y+\beta_{16} xy+\beta_{17} x^2+\beta_{18} y^2
\end{aligned} \tag{66}$$

Substituting (66) in (60), we find the constraint equations:

$$\begin{aligned}
(\alpha_2+\alpha_4 y+2\alpha_5 x) &= (\beta_2+\beta_9)+(\beta_4+2\beta_{12})y+(2\beta_5+\beta_{10})x \\
(\alpha_3+\alpha_4 x+2\alpha_6 y) &= (\beta_8+\beta_{15})+(\beta_{16}+2\beta_{11})x+(\beta_{10}+2\beta_{18})y
\end{aligned} \tag{67}$$

which can be met by setting:

$$\begin{aligned}
&\alpha_2=(\beta_2+\beta_9); \qquad \alpha_3=(\beta_8+\beta_{15}) \\
&2\alpha_5=(2\beta_{15}+\beta_{10}); \qquad 2\alpha_6=(\beta_{10}+2\beta_{18})y \\
&\beta_4=\beta_{16};\ \beta_{11}=\beta_{12};\ \alpha_4=(\beta_4+2\beta_{12})
\end{aligned} \tag{68}$$

Thus, the pressure p can be written as:

$$\begin{aligned}
p &= \alpha_1+[\alpha_2 x+\alpha_3 y+\alpha_4 xy+\alpha_5 x^2+\alpha_6 y^2] \\
&\equiv \alpha_1+[(\beta_2+\beta_9)x+(\beta_8+\beta_{15})y+(\beta_4+2\beta_{12})x \\
&\quad +\tfrac{1}{2}(2\beta_5+\beta_{10})x^2+\tfrac{1}{2}(\beta_{10}+2\beta_{18})y^2]
\end{aligned} \tag{69}$$

and σ'_{ij} as:

$$\begin{aligned}
\sigma'_{11} &= \beta_1+\beta_2 x+\beta_3 y+\beta_4 xy+\beta_5 x^2+\beta_6 y^2 \\
\sigma'_{12} &= \beta_7+\beta_8 x+\beta_9 y+\beta_{10} xy+\beta_{11}(x^2+y^2) \\
\sigma'_{22} &= \beta_{13}+\beta_{14} x+\beta_{15} y+\beta_4 xy+\beta_{17} x^2+\beta_{18} y^2
\end{aligned} \tag{70}$$

Note that there are only 16 parameters in σ'_{ij} in (70); however, to avoid confusion, the numbering of the β parameters has been left as it was in (66). Equation (69) is an example of the explicit representation of the general equation (30b) in the text. Once the element parameters β in σ'_{ij} are calculated from the element nodal velocities using equation (35b) and the constant term α_1 in the pressure variation is computed from the global finite element equations equation (40a), the polynomial variation of pressure can be computed from equation (69).

APPENDIX II: COMPARISON OF THE PRESENT MIXED METHOD WITH THE STANDARD VELOCITY-PRESSURE FORMULATION

To further clarify the present assumed 'deviatoric stress–velocity–pressure' mixed method, it is compared herein with the standard 'velocity-pressure' mixed method. Apart from the obvious differences in the variational bases for the two approaches, we discuss here specifically the finite element equations that would arise for the standard 'velocity–pressure' formulation and compare them with those for the present method as given in equations (32) and (33).

We will only consider the standard Galerkin method for the 'velocity–pressure' formulation, which has as its variational basis, equations (11)–(13) respectively. Now, equation (11) can be written (assuming that $\bar{F}_i = 0$) as:

$$\delta P + \delta N = 0 = \int_V \left\{(2\mu V_{ij} - p\,\delta_{ij})\,\delta V_{ij} - \delta p V_{kk} + \rho\left(\frac{\partial v_i}{\partial t} + v_{i,j}v_j\right)\delta v_i\right\}\mathrm{d}V - \int_{S_t} \bar{T}_i\,\delta v_i\,\mathrm{d}s \qquad (71)$$

Suppose that in each element, the velocity and pressure are approximated as:

$$v_i = B_{ir}q_r \qquad (r = 1 \ldots N_q)$$

and

$$p = A_\alpha p_\alpha \qquad (\alpha = 1 \ldots N_p) \qquad (72)$$

where v is C° continuous and is, in general, one degree higher polynomial than p. Thus, the finite element counterpart of (71) becomes:

$$\delta P + \delta N = 0 = \sum_m \left\{\int_{V_m}\left[(2\mu V_{ij} - p\,\delta_{ij})\,\delta V_{ij} - \delta p V_{kk} + \rho\left(\frac{\partial v_i}{\partial t} + v_{i,j}v_j\right)\delta v_i\right]\mathrm{d}V - \int_{S_{tm}} \bar{T}_i\,\delta v_i\,\mathrm{d}s\right\} \qquad (73)$$

Since v_i is C° continuous $\delta V_{ij} = \delta v_{(i,j)}$, upon using the divergence theorem, equation (73) becomes:

$$\begin{aligned} 0 = \sum_m \Bigg\{ & \int_{V_m} [-(2\mu V_{ij} - p\,\delta_{ij})_{,j}]\,\delta v_i\,\mathrm{d}V \\ & + \int_{\partial V_m} n_j(2\mu V_{ij} - p\,\delta_{ij})\,\delta v_i\,\mathrm{d}s - \int_{S_{tm}} \bar{T}_i\,\delta v_i\,\mathrm{d}s \\ & + \int_{V_m} \rho\left(\frac{\partial v_i}{\partial t} + v_{i,j}v_j\right)\delta v_i\,\mathrm{d}V - \int_{V_m} \delta p V_{kk}\,\mathrm{d}V \Bigg\} \end{aligned} \qquad (74)$$

It is seen that the first four terms of equation (74) lead to the weighted residual form of the finite element momentum balance equation and the last term to that of the incompressibility constraint. In order to understand the physical meaning of the first 4 terms of (74), we note once again that in this method v_i are subject to nodal connectivity through q_r, and B_{ir} of (72) once again have only a local support. Thus, once again consider a 'patch' with several elements meeting at a node. A virtual velocity at this node affects only the surrounding elements. Let δp_r be the virtual velocity at this node. Let the elements in the patch be N_p. Then the first four terms of (74) imply that:

$$\begin{aligned} 0 = \sum_{\substack{\mathrm{ele} \\ e=1}}^{N_p} \Bigg\{\Bigg\langle & \int_{V_e} -[\mu(B_{im,j} + B_{jm,i})q_m - A_\alpha p_\alpha\,\delta_{ij}]_{,j}B_{ir}\,\mathrm{d}V \\ & + \int_{\partial V_e} n_j[\mu(B_{im,j} + B_{jm,i})q_m - A_\alpha p_\alpha\,\delta_{ij}]B_{ir}\,\mathrm{d}s \\ & - \int_{S_{te}} \bar{T}_i B_{ir}\,\mathrm{d}s + \int_{V_e} \rho\left(B_{ir}B_{is}\frac{\partial q_s}{\partial t} + B_{ir}B_{it,j}B_{js}q_t q_s\right)\mathrm{d}V\Bigg\rangle\,\delta q_r\Bigg\} \end{aligned} \qquad (75)$$

Note further that, if the deviatoric stresses computed from the assumed v_i are labeled as $(\sigma'_{ij})^*$, we have:

$$(\sigma'_{ij})^* = 2\mu V_{ij} = \mu[B_{im,j} + B_{jm,i}]q_m \tag{76}$$

Note also that since the assumed velocities in the standard velocity pressure mixed formulation are simply C° continuous functions with no further constraints, it is seen that, in general,

$$[(\sigma'_{ij})^* - p\,\delta_{ij}]_{,j} \neq 0 \quad \text{in } V_e \tag{77a}$$

and

$$[(\sigma'_{ij})^* - p\,\delta_{ij}]n_j^+ + [(\sigma'_{ij})^* - p\,\delta_{ij}]n_j^- \neq 0 \quad \text{at } \partial V_c \tag{77b}$$

Note, on the other hand, that in the present assumed 'deviatoric stress–pressure–velocity' mixed formulation the directly assumed σ'_{ij} and p are such that

$$[\sigma'_{ij} - p\,\delta_{ij}]_{,j} = 0 \quad \text{in } V_e \tag{78a}$$

but

$$\{[\sigma'_{ij} - p\,\delta_{ij}]n_j\}^+ + \{[\sigma'_{ij} - p\,\delta_{ij}]n_j\}^- \neq 0 \quad \text{at } \partial V_e \tag{78b}$$

Comparing finite element weighted residual equation (75) for the standard velocity–pressure formulation with the corresponding equation (33) for the present 'deviatoric stress–velocity–pressure' formulations, the distinctions between the two methods can be immediately noted. Firstly, both (75) and (33) are essentially similar, except that the counterpart of the first term of equation (75) (which is not zero owing to equation (77a)) vanishes in equation (33) owing to the *a priori* constraint (78a). From a weighted residual point of view, this is, if anything, not a shortcoming. In addition, the constraint (78a) of the present method enables all but the constant term in the pressure variation in each element to be expressed in terms of the undetermined parameters in the assumed σ'_{ij}. This is decidely an advantage compared to the standard velocity–pressure formulation wherein all the parameters p_α of equation (72) in each element are retained as solution variables in the global finite element system of equations. Also, from equation (75), it is seen that the linear viscous 'stiffness' matrix for the standard velocity pressure approach comes from the first two terms of equation (75). On the other hand, in the present approach, the linear viscous 'stiffness' matrix comes from only the first term of equation (33) when β_n are expressed in terms of q_r as in equation (35b).

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